

# When does the heat equation have a solution with a sequence of similar level sets?

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## Abstract

In this paper we consider the overdetermined Cauchy problem for the heat equation. We prove that if the problem has a nontrivial nonnegative solution with a certain sequence of similar level sets, then the solution must be radially symmetric.

## 1 Introduction

Consider the unique bounded solution  $u = u(x, t)$  of the Cauchy problem for the heat equation:

$$(1.1) \quad \partial_t u = \Delta u \quad \text{in} \quad \mathbb{R}^N \times (0, +\infty), \quad u = g \geq 0 \quad \text{on} \quad \mathbb{R}^N \times \{0\},$$

where  $N \geq 1$  and  $g$  is a nontrivial bounded continuous function whose support is compact. For the initial data  $g$ , we denote by  $G_0$  the support of  $g$ , namely  $G_0 = \text{spt}(g)$ .

For problem (1.1), it is well known that if  $g$  is radially symmetric, then the solution  $u$  of (1.1) must be radially symmetric. The problem determining the shape of solutions by using a geometric information of solutions is an interesting one in the study of qualitative properties of partial differential equations. In [7, Corollary 3.2, p.4829], problem (1.1), where  $g$  is replaced by a characteristic function of a bounded open set, is considered, and it is shown that if there exists a non-empty stationary isothermic surface of  $u$ , then  $u$  must be radially symmetric. In this paper we consider another type of overdetermination. Precisely we consider the Cauchy problem (1.1) which have a solution with a certain sequence of similar level sets, and prove the following.

**Theorem 1.1** *Let  $N \geq 1$  and  $G_0$  be a compact set containing the origin. Suppose that  $\Omega$  is a bounded domain with  $C^1$  boundary  $\partial\Omega$  such that  $G_0 \subset \Omega$ . Assume that the solution  $u$  of (1.1) satisfies the following condition:*

$$(C) \quad \left\{ \begin{array}{l} \text{there exist two sequences of positive numbers } \{t_n\}_{n=1}^{\infty} \text{ and } \{a_n\}_{n=1}^{\infty} \text{ such that} \\ t_n \uparrow \infty \text{ as } n \uparrow \infty \text{ and } u(t_n x, t_n) = a_n \text{ for all } n \in \mathbb{N} \text{ and } x \in \partial\Omega. \end{array} \right.$$

*Then  $u$  must be radially symmetric with respect to the origin.*

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The proof of Theorem 1.1 consists of two steps. In the first step, by using condition (C) and the monotonicity of solutions on some exterior domain (see (2.3)) we see that  $\partial\Omega$  is a sphere with center the origin (see Lemma 2.1 and Proposition 2.1), and we prove the radially symmetric property of the solution in the second step.

Our argument in the first step is also applicable to some elliptic boundary value problems. As an example, we consider the following boundary value problem for some fully nonlinear elliptic equation. Let  $u \in C^1(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$  be the unique viscosity solution of

$$(1.2) \quad \begin{cases} F(D^2u, Du, u) = 0 & \text{in } \mathcal{D}, \\ u > 0 & \text{in } \mathcal{D}, \\ u = 1 & \text{on } \partial\Omega, \\ u \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases}$$

where  $N \geq 1$ ,  $\Omega \subset \mathbb{R}^N$  is a bounded domain satisfying  $0 \in \Omega$  with  $C^1$  boundary  $\partial\Omega$ , and  $\mathcal{D} = \mathbb{R}^N \setminus \overline{\Omega}$  is also a domain. Here the nonlinearity  $F$  satisfies the following:

(H1) (Regularity)  $F$  is a continuous function defined on  $\mathcal{S}^N(\mathbb{R}) \times \mathbb{R}^N \times \mathbb{R}$ , where  $\mathcal{S}^N$  denotes the space of  $N \times N$  symmetric (real) matrices. Furthermore, for any  $R > 0$  there exists a positive constant  $C_1$  such that

$$|F(M, p, u_1) - F(N, q, u_2)| \leq C_1\{|M - N| + |p - q| + |u_1 - u_2|\}$$

for all  $M, N \in \mathcal{S}^N(\mathbb{R})$ ,  $p, q \in \mathbb{R}^N$ , and  $u_1, u_2 \in [-R, R]$ .

(H2) (Ellipticity) There exists a constant  $C_2 > 0$  such that

$$F(M + N, p, u) - F(M, p, u) \geq C_2 \text{Tr}(N)$$

for all  $M, N \in \mathcal{S}^N(\mathbb{R})$  with  $N \geq 0$ ,  $p \in \mathbb{R}^N$ , and  $u \in \mathbb{R}$ .

(H3) (Symmetry) For any  $M \in \mathcal{S}^N(\mathbb{R})$ ,  $A \in \mathcal{O}^N(\mathbb{R})$ ,  $p \in \mathbb{R}^N$ , and  $u \in (0, \infty)$ ,

$$F(M, p, u) = F({}^t A M A, {}^t A p, u),$$

where  $\mathcal{O}^N(\mathbb{R})$  denotes the set of  $N$ -dimensional orthogonal matrices and  ${}^t A$  denotes the transpose of  $A \in \mathcal{O}^N(\mathbb{R})$ .

(H4) (Homogeneity) There exists a some constant  $\beta < 0$  such that, for any  $\mu > 1$ , the function

$$(1.3) \quad u_\mu(x) := \mu^\beta u(\mu^{-1}x)$$

is a solution of problem (1.2), where  $\Omega$  is replaced by

$$\Omega_\mu = \{\mu x \in \mathbb{R}^N : x \in \Omega\}$$

and where the boundary condition  $u = 1$  on  $\partial\Omega$  is replaced by

$$u_\mu(x) = \mu^\beta \quad \text{on } \partial\Omega_\mu.$$

Then the following holds.

**Theorem 1.2** *Suppose that  $F$  satisfies (H1)–(H4), and  $F$  is nonincreasing in  $u > 0$ . Let  $u \in C^1(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$  be a viscosity solution of (1.2). Assume that there exists a constant  $\lambda > 1$  such that  $\overline{\Omega} \subset \Omega_\lambda$  and*

$$(1.4) \quad u(x) = \lambda^\beta \quad \text{for all } x \in \partial\Omega_\lambda.$$

*Then  $\partial\Omega$  is a sphere with center the origin and  $u$  must be radially symmetric.*

**Remark 1.1** *In [4] Enache and the second author of the present paper studied some overdetermined boundary value problem for fully nonlinear elliptic problem in a bounded domain  $\Omega$ . They proved that if there exists a constant  $\alpha < 0$  such that  $u(x) = \alpha$  on  $\partial\Omega_\lambda$  with  $\lambda \in (0, 1)$ , then  $\Omega$  must be the interior of an  $N$ -dimensional ellipsoid. See [4, Theorem 2.1].*

The paper is organized as follows. In Section 2 we give some preliminary proposition, and prove the key lemma of this paper. Applying this lemma we prove main theorems in Section 3. In Section 4 we give two remarks on condition (C) concerning a sequence of similar level sets.

## 2 Preliminary

We prepare several notations. For each  $r > 0$  and  $z \in \mathbb{R}^N$ , denote by  $B_r(z)$  the open ball in  $\mathbb{R}^N$  with radius  $r$  and center  $z$ . For each  $l \in \partial B_1(0)$ , let  $\Pi_l \subset \mathbb{R}^N$  be the hyperplane with normal  $l$  and  $0 \in \Pi_l$ , that is,

$$\Pi_l = \{x \in \mathbb{R}^N : x \cdot l = 0\}.$$

For each  $C^1$  domain  $\Omega \subset \mathbb{R}^N$ ,  $T_p(\partial\Omega)$  denotes the tangent space of  $\partial\Omega$  at  $p \in \partial\Omega$ .

We first prove the following proposition:

**Proposition 2.1** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded  $C^1$  domain containing the origin. If*

$$(2.1) \quad \{p \in \partial\Omega \mid T_p(\partial\Omega) \ni l\} = \Pi_l \cap \partial\Omega \quad \text{for every } l \in \partial B_1(0),$$

*then  $\partial\Omega$  is a sphere with center the origin.*

**Proof.** Since  $\Omega$  is a bounded  $C^1$  domain,  $\partial\Omega$  has finitely many connected components, and each component is a  $C^1$  closed hypersurface embedded in  $\mathbb{R}^N$ . Let  $\Gamma$  be a component of  $\partial\Omega$  and  $p \in \Gamma$ . For any  $l \in \partial B_1(0)$  with  $l \perp p$ , by (2.1) we can take  $\Pi_l$  with  $p \in \Pi_l$ , and obtain  $l \in T_p(\Gamma)$ . This implies that, for any  $p \in \Gamma$ , we have

$$(2.2) \quad p \perp T_p(\Gamma).$$

Let  $p = p(t)$  be a regular curve on  $\partial\Omega$ . Then, by (2.2) we obtain

$$p(t) \perp \frac{d}{dt}p(t) \quad \text{for all } t,$$

namely

$$\frac{d}{dt}(|p(t)|^2) = 0 \quad \text{for all } t.$$

Therefore we see that there exists a positive constant  $C$  such that  $|p(t)| = C$  for all  $t$ . This implies that  $\Gamma$  is a sphere with center the origin. Therefore, since  $\Omega$  is a domain containing the origin, we see that  $\partial\Omega$  must be a sphere with center the origin, and Proposition 2.1 follows.  $\square$

Next we prove the key lemma of the proof of Theorems 1.1 and 1.2.

**Lemma 2.1** *Let  $u \in C^1(\mathbb{R}^N \times (0, \infty))$ , and let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  containing the origin. Suppose that there exists a half space  $H$  of  $\mathbb{R}^N$  including  $\overline{\Omega}$  such that*

$$(2.3) \quad \frac{\partial u}{\partial l} < 0 \quad \text{in} \quad (\mathbb{R}^N \setminus H) \times (0, \infty),$$

where  $l$  is the outer unit normal vector to  $\partial H$  and suppose the following condition:

$$(C) \quad \begin{cases} \text{there exist two sequences of positive numbers } \{t_n\}_{n=1}^\infty \text{ and } \{a_n\}_{n=1}^\infty \text{ such that} \\ t_n \uparrow \infty \text{ as } n \uparrow \infty, \text{ and } u(t_n x, t_n) = a_n \text{ for all } n \in \mathbb{N} \text{ and } x \in \partial\Omega. \end{cases}$$

If  $p \in \partial\Omega$  and  $l \in T_p(\partial\Omega)$ , then  $p \cdot l \leq 0$ .

**Proof.** Without loss of generality, set  $l = (1, 0, \dots, 0)$ . Then, since  $0 \in \Omega$  and  $\overline{\Omega} \subset H$ , there exists a positive constant  $\lambda$  satisfying  $H = \{x \in \mathbb{R}^N : x_1 < \lambda\}$ . Suppose that there exists a point  $p \in \partial\Omega$  such that

$$p \cdot l = p_1 > 0 \text{ and } l \in T_p(\partial\Omega).$$

Hence, by condition (C) we have

$$(2.4) \quad \frac{\partial u}{\partial x_1}(t_n p, t_n) = 0$$

for all  $n \in \mathbb{N}$ . Since  $p_1 > 0$  and  $t_n \uparrow \infty$  as  $n \uparrow \infty$ , by (2.4) we see that there exists a sufficiently large number  $n_*$  such that  $t_{n_*} p_1 > \lambda$  and

$$\frac{\partial u}{\partial x_1}(t_{n_*} p, t_{n_*}) = 0,$$

which contradicts (2.3).  $\square$

### 3 Proof of Theorems

The purpose of this section is to prove Theorems 1.1 and 1.2. We first prove Theorem 1.2.

**Proof of Theorem 1.2.** By (1.3), (1.4) and the uniqueness of the solution of (1.2), we see that

$$u(x) = \lambda^\beta u(\lambda^{-1}x) \quad \text{for all } x \in \Omega_\lambda.$$

Therefore, setting

$$t_n = \lambda^n \text{ and } a_n = \lambda^{\beta(n+1)} \text{ for every } n \in \mathbb{N}$$

yields that the solution  $u$  of (1.2) satisfies

$$t_n \uparrow \infty \text{ as } n \uparrow \infty, \text{ and } u(t_n x) = a_n \text{ for all } n \in \mathbb{N} \text{ and } x \in \partial\Omega_\lambda.$$

By the method of moving planes, the maximum principle (see [2] and [3, Proposition 2.6]), and Hopf's boundary point lemma, we see that, for every direction  $l \in \partial B_1(0)$ , there exists a half space  $H$  of  $\mathbb{R}^N$  including  $\overline{\Omega}$  and having the outer unit normal vector  $l$  of  $\partial H$  such that the solution of (1.2) satisfies (2.3). Therefore we can apply Lemma 2.1 to every direction  $l \in \partial B_1(0)$  and conclude that  $\Omega$  satisfies (2.1). Hence by Proposition 2.1  $\partial\Omega$  must be a sphere with center the origin. Therefore, since for every  $A \in \mathcal{O}^N(\mathbb{R})$  the function  $u(Ax)$  also satisfies (1.2) by (H3), then, by the uniqueness of the solution of (1.2),  $u(x) \equiv u(Ax)$  and hence  $u$  must be radially symmetric. The proof of Theorem 1.2 is complete.  $\square$

Next we prove Theorem 1.1. Let  $H$  be an arbitrary half space of  $\mathbb{R}^N$  including  $G_0$ . Then, by the method of moving planes (see [6], for example), the maximum principle and Hopf's boundary point lemma we have

$$(3.1) \quad \frac{\partial u}{\partial l} < 0 \quad \text{in} \quad (\mathbb{R}^N \setminus H) \times (0, \infty),$$

where  $l$  is the outer unit normal vector to  $\partial H$ . Therefore, by (3.1) and condition (C) we can use Lemma 2.1 and hence by Proposition 2.1  $\partial\Omega$  must be a sphere with center the origin for  $N \geq 2$ .

We first prove Theorem 1.1 with  $N = 1$ .

**Proof of Theorem 1.1 for  $N = 1$ .** Since  $0 \in G_0 \subset \Omega$ , we can set  $\Omega = (a, b)$  for some  $a < 0 < b$ . Then it follows from condition (C) that

$$(3.2) \quad u(t_n a, t_n) = u(t_n b, t_n) (= a_n) \text{ for every } n \in \mathbb{N}.$$

Let us show that  $a + b = 0$ . Suppose that  $a + b > 0$ . Then, since  $t_n \uparrow \infty$  as  $n \uparrow \infty$ , there exists  $m \in \mathbb{N}$  such that

$$(3.3) \quad \frac{t_m(a+b)}{2} > b.$$

Consider the function  $w = w(x, t)$  defined by

$$w(x, t) = u(x, t) - u(t_m a + t_m b - x, t).$$

Then we have from (3.3) the following:

$$\begin{aligned} \partial_t w &= \partial_x^2 w && \text{in } (t_m(a+b)/2, +\infty) \times (0, +\infty), \\ w &= 0 && \text{on } \{t_m(a+b)/2\} \times (0, +\infty), \\ w &\leq 0 \text{ and } w \not\equiv 0 && \text{on } (t_m(a+b)/2, +\infty) \times \{0\}. \end{aligned}$$

Thus it follows from the strong maximum principle that

$$w < 0 \quad \text{in } (t_m(a+b)/2, +\infty) \times (0, +\infty),$$

which contradicts the fact that  $w(t_m b, t_m) = 0$  because of (3.2). Therefore, we conclude that  $a + b \leq 0$ . By the same argument we also conclude that  $a + b \geq 0$ .

Here we can put  $\Omega = (-b, b)$ . For  $(x, t) \in [0, \infty) \times [0, \infty)$ , consider the functions  $v = v(x, t)$  and  $v_0 = v_0(x)$  defined by

$$v(x, t) = u(x, t) - u(-x, t) \quad \text{and} \quad v_0(x) = g(x) - g(-x).$$

It suffices to prove

$$(3.4) \quad v_0(x) = 0 \quad \text{for all } x \in [0, \infty).$$

Indeed, if (3.4) holds, then  $g(x) = g(|x|)$  for all  $x \in \mathbb{R}$ . This together with the uniqueness of the solution yields the conclusion of Theorem 1.1 with  $N = 1$ .

Since  $G_0 \subset \Omega = (-b, b)$ , we see that  $\text{spt}(v_0) \subset (-b, b)$ . This implies that  $v$  satisfies

$$(3.5) \quad v(x, t) = (4\pi t)^{-\frac{1}{2}} \int_{-b}^b e^{-\frac{(x-y)^2}{4t}} v_0(y) dy$$

for all  $(x, t) \in \mathbb{R} \times (0, +\infty)$ . Furthermore, by (3.2) with  $a = -b$ , we see that

$$v(t_n b, t_n) = 0 \quad \text{for every } n \in \mathbb{N}.$$

This together with (3.5) yields

$$(3.6) \quad \int_{-b}^b v_0(y) e^{\frac{by}{2}} e^{-\frac{y^2}{4t_n}} dy = 0 \quad \text{for every } n \in \mathbb{N}.$$

Put

$$(3.7) \quad w_0(y) = v_0(y) e^{\frac{by}{2}}.$$

Then, by (3.6) we have

$$\int_0^{b^2} \frac{w_0(\sqrt{s}) + w_0(-\sqrt{s})}{2\sqrt{s}} e^{-\frac{s}{4t_n}} ds = 0 \quad \text{for every } n \in \mathbb{N}.$$

Since  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ , by the analyticity of the exponential function we obtain

$$\int_0^{b^2} \frac{w_0(\sqrt{s}) + w_0(-\sqrt{s})}{2\sqrt{s}} e^{-\lambda s} ds = 0, \quad \lambda \in \mathbb{R}.$$

This together with the injectivity of the Laplace transform yields

$$(3.8) \quad w_0(\sqrt{s}) + w_0(-\sqrt{s}) = 0 \quad \text{for all } s > 0.$$

By (3.7) and (3.8) we have

$$v_0(\sqrt{s})e^{\frac{b\sqrt{s}}{2}} + v_0(-\sqrt{s})e^{-\frac{b\sqrt{s}}{2}} = 0 \quad \text{for all } s > 0.$$

This implies that  $v_0(\sqrt{s}) = 0$  for all  $s > 0$ . Thus we have (3.4), and Theorem 1.1 with  $N = 1$  follows.  $\square$

Next we prove Theorem 1.1 for the case  $N \geq 2$ . Before beginning the proof we recall the following lemma, which follows from the Funk-Hecke formula (see [1, Theorem 2.22, p. 36] or [8, Theorem 6, p. 20]) and Rodrigues' formula (see [1, Theorem 2.23, p. 37] or [8, Theorem 5, p. 17]).

**Lemma 3.1** *Let  $L \neq 0$  be a real constant. For  $f \in L^2(S^{N-1})$ , set*

$$\mathcal{L}f(\omega) = \int_{S^{N-1}} e^{L\alpha \cdot \omega} f(\alpha) d\sigma(\alpha) \quad \text{for every } \omega \in S^{N-1},$$

where  $d\sigma(\alpha)$  denotes the area element of the  $(N-1)$ -dimensional unit sphere  $S^{N-1}$  in  $\mathbb{R}^N$ . Then the set  $\{\mathcal{L}f : f \in L^2(S^{N-1})\}$  is dense in  $L^2(S^{N-1})$ .

**Proof.** Let  $p = p(x)$  be an arbitrary harmonic homogeneous polynomial of degree  $k \geq 0$  in  $\mathbb{R}^N$ . Then it follows from the Funk-Hecke formula that

$$\mathcal{L}p(\omega) = \lambda p(\omega) \quad \text{for every } \omega \in S^{N-1} \quad \text{and } \lambda = |S^{N-2}| \int_{-1}^1 e^{Lt} P_k(t) (1-t^2)^{\frac{N-3}{2}} dt,$$

where  $|S^{N-2}|$  denotes the volume of the  $(N-2)$ -dimensional unit sphere in  $\mathbb{R}^{N-1}$  and  $P_k(t)$  denotes the Legendre polynomial of degree  $k$  in  $\mathbb{R}^N$ . Moreover, Rodrigues' formula gives

$$P_k(t) = (-1)^k \frac{\Gamma(\frac{N-1}{2})}{2^k \Gamma(k + \frac{N-1}{2})} (1-t^2)^{\frac{3-N}{2}} \left( \frac{d}{dt} \right)^k (1-t^2)^{k + \frac{N-3}{2}}.$$

Therefore, integrating by parts  $k$  times on the definition of the number  $\lambda$  yields that

$$\lambda = |S^{N-2}| \frac{\Gamma(\frac{N-1}{2})}{2^k \Gamma(k + \frac{N-1}{2})} L^k \int_{-1}^1 e^{Lt} (1-t^2)^{\frac{N-3}{2}} dt \neq 0.$$

This implies that the linear space  $\{\mathcal{L}f : f \in L^2(S^{N-1})\}$  contains all the spherical harmonics because any spherical harmonic is given by restricting a harmonic homogeneous polynomial onto  $S^{N-1}$ . Therefore, the conclusion holds true.  $\square$

Now we are ready to prove Theorem 1.1 for the case  $N \geq 2$ .

**Proof of Theorem 1.1 for  $N \geq 2$ .** First of all, since we already know that  $\partial\Omega$  is a sphere with center the origin, there exists a constant  $R > 0$  such that  $\Omega = B_R(0)$ . Take  $A \in \mathcal{O}^N(\mathbb{R})$  arbitrarily. Then, it follows from (C) that

$$(3.9) \quad u(t_n x, t_n) - u(t_n A x, t_n) = 0, \quad x \in \partial B_R(0),$$

for all  $n \in \mathbb{N}$ . Since  $G_0 \subset B_R(0)$ , by (3.9) we have

$$\int_{|y| \leq R} e^{\frac{x \cdot y}{2}} e^{-\frac{|y|^2}{4t_n}} (g(y) - g(Ay)) dy = 0, \quad x \in \partial B_R(0),$$

for all  $n \in \mathbb{N}$ . This, together with the analyticity of the exponential function and the fact that  $t_n \rightarrow \infty$  as  $n \rightarrow \infty$ , implies that

$$(3.10) \quad \int_{|y| \leq R} e^{\frac{x \cdot y}{2}} e^{-s|y|^2} (g(y) - g(Ay)) dy = 0, \quad x \in \partial B_R(0),$$

for all  $s \in \mathbb{R}$ . By setting  $x = R\alpha$  for  $\alpha \in S^{N-1}(= \partial B_1(0))$ , we have from (3.10) that

$$\int_0^R r^{N-1} e^{-sr^2} \int_{S^{N-1}} e^{\frac{Rr}{2} \alpha \cdot \omega} (g(r\omega) - g(rA\omega)) d\sigma(\omega) dr = 0, \quad \alpha \in S^{N-1},$$

for all  $s \in \mathbb{R}$ . This together with the injectivity of the Laplace transform yields

$$(3.11) \quad \int_{S^{N-1}} e^{\frac{Rr}{2} \alpha \cdot \omega} (g(r\omega) - g(rA\omega)) d\sigma(\omega) = 0, \quad \alpha \in S^{N-1},$$

for all  $r \geq 0$ . Let  $f \in L^2(S^{N-1})$  and fix  $r > 0$ . Then, by (3.11) we obtain

$$\int_{S^{N-1}} \int_{S^{N-1}} e^{\frac{Rr}{2} \alpha \cdot \omega} (g(r\omega) - g(rA\omega)) f(\alpha) d\sigma(\omega) d\sigma(\alpha) = 0.$$

Thus, setting  $L = \frac{Rr}{2}$  in Lemma 3.1 yields that

$$\int_{S^{N-1}} \mathcal{L}f(\omega) (g(r\omega) - g(rA\omega)) d\sigma(\omega) = 0 \quad \text{for every } f \in L^2(S^{N-1}).$$

Then, by Lemma 3.1 we have

$$g(r\omega) = g(rA\omega)$$

for all  $(r, \omega) \in (0, R] \times S^{N-1}$ . Since  $A \in \mathcal{O}^N(\mathbb{R})$  is arbitrary, we see that the initial data  $g$  is radially symmetric with respect to the origin. This together with the uniqueness of the solution yields the conclusion of Theorem 1.1 for  $N \geq 2$ .  $\square$

## 4 Remarks on condition (C)

In this last section we give two remarks on condition (C) concerning a sequence of similar level sets.

**Remark 4.1** *There exists a solution of (1.1) with  $N = 3$  which is not radially symmetric even if it has a sequence of similar level sets.*

*Let  $a$  be a positive constant and  $v_0 \in C_0^\infty(\mathbb{R})$  be a nonnegative even function satisfying*

$$(4.1) \quad \text{spt}(v_0) = [-a, a] \quad \text{and} \quad v'_0 < 0 \quad \text{if} \quad s \in (0, a).$$



Put

$$(4.2) \quad v(s, t) = (4\pi t)^{-\frac{1}{2}} \int_{-a}^a e^{-\frac{(s-\mu)^2}{4t}} v_0(\mu) d\mu, \quad w(s, t) = \frac{\partial^3}{\partial s^3} v(s, t),$$

for  $(s, t) \in \mathbb{R} \times (0, \infty)$ . Then, since  $v$  is a even function in  $s$ ,  $w$  is a odd function in  $s$ . Furthermore, we set  $r = |x|$  for  $x \in \mathbb{R}^3$  and put

$$(4.3) \quad f(r, t) = \frac{\partial}{\partial r} \left( \frac{w(r, t)}{r} \right) = \left( r \frac{\partial w}{\partial r} - w \right) r^{-2}.$$

Then we have the following lemma:

**Lemma 4.1** *Let  $f$  be the function given in (4.3). Then there exists a positive function  $r(t)$  for  $t > 0$  such that*

$$f(r(t), t) = 0, \quad r(t) = O(t^{\frac{1}{2}}) \quad \text{as } t \rightarrow \infty.$$

**Proof.** By (4.1) and (4.2) we can take three positive functions  $r_2(t)$ ,  $r_3(t)$  and  $r_4(t)$  for  $t > 0$  such that  $r_2(t) < r_3(t) < r_4(t)$  and

$$(4.4) \quad \frac{\partial^3}{\partial s^3} v(r_3(t), t) = 0, \quad \frac{\partial^3}{\partial s^3} v(s, t) < 0 \quad \text{if } s > r_3(t),$$

$$(4.5) \quad \frac{\partial^4}{\partial s^4} v(r_2(t), t) = \frac{\partial^4}{\partial s^4} v(r_4(t), t) = 0,$$

$$(4.6) \quad \frac{\partial^4}{\partial s^4} v(s, t) < 0 \quad \text{if } r_2(t) < s < r_4(t), \quad \text{and } \frac{\partial^4}{\partial s^4} v(s, t) > 0 \quad \text{for } s > r_4(t),$$

for all  $t > 0$ . Put

$$h(s, t) = s^2 f(s, t) = s \frac{\partial^4 v}{\partial s^4} - \frac{\partial^3 v}{\partial s^3}$$

Then, since  $h(s, t) > 0$  for  $s \geq r_4(t)$  and  $h(s, t) < 0$  for  $s \in [r_2(t), r_3(t)]$ , by applying the intermediate value theorem, we can take a positive function  $r(t)$  for  $t > 0$  such that

$$(4.7) \quad f(r(t), t) = 0, \quad r_3(t) < r(t) < r_4(t).$$

On the other hands, by (4.2) we obtain

$$(4.8) \quad (4\pi t)^{\frac{1}{2}} \frac{\partial^3 v}{\partial s^3} = -\frac{1}{8t^3} \int_{-a}^a (s - \mu) \{-6t + (s - \mu)^2\} e^{-\frac{(s-\mu)^2}{4t}} v_0(\mu) d\mu,$$

$$(4.9) \quad (4\pi t)^{\frac{1}{2}} \frac{\partial^4 v}{\partial s^4} = \frac{1}{16t^4} \int_{-a}^a \{12t^2 - 12t(s - \mu)^2 + (s - \mu)^4\} e^{-\frac{(s-\mu)^2}{4t}} v_0(\mu) d\mu$$

$$> \frac{1}{16t^4} \int_{-a}^a (s - \mu)^2 \{(s - \mu)^2 - 12t\} e^{-\frac{(s-\mu)^2}{4t}} v_0(\mu) d\mu.$$

for all  $(s, t) \in \mathbb{R} \times (0, \infty)$ . By (4.8) we see that

$$\frac{\partial^3 v}{\partial s^3} < 0 \quad \text{for } s > a + \sqrt{6t},$$

$$\frac{\partial^3 v}{\partial s^3} > 0 \quad \text{for } a < s < \sqrt{5t} - a \quad \text{with } t > \frac{4a^2}{5}.$$

These together with (4.4) yield

$$(4.10) \quad \sqrt{5t} - a \leq r_3(t) \leq a + \sqrt{6t} \quad \text{for } t > \frac{4a^2}{5}.$$

Furthermore, by (4.9) we have

$$\frac{\partial^4 v}{\partial s^4} > 0 \quad \text{for } s > a + \sqrt{12t}.$$

This together with (4.5) implies that

$$(4.11) \quad r_4(t) \leq a + \sqrt{12t}.$$

Combining (4.7), (4.10) and (4.11), we obtain

$$\sqrt{5t} - a \leq r_3(t) < r(t) < r_4(t) \leq a + \sqrt{12t}$$

for all  $t > (4a^2)/5$ . This implies that  $r(t) = O(t^{\frac{1}{2}})$  as  $t \rightarrow \infty$ ; thus Lemma 4.1 follows.  $\square$

On the other hand, by (4.1) and (4.3) we can take a nonnegative radially symmetric function  $\psi(|x|) \in C_0^\infty(\mathbb{R}^3)$  such that

$$\psi(|x|) + f(|x|, 0) \frac{x_1}{|x|} \geq 0$$

for all  $x \in \mathbb{R}^3$ . We set

$$(4.12) \quad u(x, t) = (4\pi t)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\frac{|x-y|^2}{4t}} \psi(|y|) dy + f(|x|, t) \frac{x_1}{|x|} := u_{rad}(|x|, t) + f(|x|, t) \frac{x_1}{|x|}.$$

Then the function  $u$  is a solution of (1.1) with  $N = 3$  and

$$g(x) = \psi(|x|) + f(|x|, 0) \frac{x_1}{|x|}.$$

By Lemma 4.1 and (4.12) we see that, if  $|x| = r(t)$ , then there exists a function  $c(t)$  such that

$$u(x, t) = u_{rad}(r(t), t) = c(t).$$

This implies that the solution  $u$  is not radially symmetric even if it has a sequence of similar level sets.

**Remark 4.2** Instead of (C), suppose that there exists a function  $a = a(t)$  for  $t > 0$  such that

$$u((1+t)x, t) = a(t)$$

for all  $(x, t) \in \partial\Omega \times (0, \infty)$ . Then we can use the maximum principle and the unique continuation theorem and get the same conclusion of Theorem 1.1.

Indeed, as in the proof of Theorem 1.1, by the method of moving planes, the maximum principle, Hopf's boundary point lemma, Proposition 2.1 and Lemma 2.1, we see that  $\partial\Omega$  must be a sphere with center the origin for  $N \geq 2$ . Say  $\partial\Omega = \partial B_R(0)$  for some  $R > 0$ . Take  $A \in \mathcal{O}^N(\mathbb{R})$  arbitrarily. Then

$$u((1+t)x, t) - u((1+t)Ax, t) = 0, \quad x \in \partial B_R(0), \quad t > 0.$$

Since  $u((1+0)x, 0) - u((1+0)Ax, 0) = 0$  if  $x \notin B_R(0)$ , by the maximum principle we get

$$u(x, t) - u(Ax, t) = 0 \quad \text{if } |x| > (1+t)R.$$

Hence it follows from the unique continuation theorem (see [5]) that  $u(x, t) - u(Ax, t)$  equals zero identically, which gives the conclusion.

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